LOCAL INDEPENDENCE OF FRACTIONAL BROWNIAN MOTION

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ABSTRACT. Let $\sigma_{(t,t')}$ be the sigma-algebra generated by the differences $X_s - X_{s'}$ with $s, s' \in (t, t')$, where $(X_t)_{-\infty<t<\infty}$ is the fractional Brownian motion with Hurst index $H \in (0,1)$. We prove that for any two distinct timepoints t_1 and t_2 the sigma-algebras $\sigma_{(t_1-\varepsilon,t_1+\varepsilon)}$ and $\sigma_{(t_2-\varepsilon,t_2+\varepsilon)}$ are asymptotically independent as $\varepsilon \searrow 0$. We show this in the strong sense that Shannon's mutual information between the two σ-algebras tends to zero as $\varepsilon \searrow 0$. Some generalizations and quantitative estimates are provided also.

1. INTRODUCTION

Let $X = (X_t)_{-\infty<t<\infty}$ be the standard fractional Brownian motion (FBM) with Hurst index $H \in (0,1)$. Thus, X_t is a centered Gaussian process with stationary increments and variance function $\mathbf{E}\, X_t^2 = |t|^{2H}$ (see, e.g., [7, 11]). The parameter value $H = \frac{1}{2}$ yields the standard Brownian motion. FBM is a H-self-similar process, that is, $(X_{at}) \overset{(d)}{=} (a^H X_t)$, where $\overset{(d)}{=}$ stands for the equality of finite-dimensional distributions.

When $H \neq \frac{1}{2}$, the increments of X on disjoint time intervals are always dependent — negatively correlated for $H < \frac{1}{2}$ and positively correlated for $H > \frac{1}{2}$. Moreover, when $H > \frac{1}{2}$, the sequence X_1, $X_2 - X_1$, $X_3 - X_2, \ldots$ is long-range dependent. i.e. $\sum_{i=1}^{\infty} \mathbf{E}\, X_1(X_{i+1} - X_i) = \infty$ (see [2, 11]). FBMs with $H > \frac{1}{2}$ are often used in applications as a mathematical model for far-reaching dependence.

However, as we show in this paper, 'small and distant' events in FBMs are nevertheless asymptotically independent. This holds both as asymptotic orthogonality of the Gaussian subspaces generated by the processes $(X_t)_{|t|<1}$ and $(X_{n+t} - X_n)_{|t|<1}$ as $n \to \infty$, and in the stronger sense that the mutual (Shannon) information $I((X_t)_{|t|<1} : (X_{n+t} - X_n)_{|t|<1})$ is finite and decays to zero as $n \to \infty$. By self-similarity, this is equivalent to considering the increment processes around two fixed timepoints, $(X_{s+u} - X_s)_{|u|<\varepsilon}$ and $(X_{t+u} - X_t)_{|u|<\varepsilon}$, as $\varepsilon \searrow 0$. We propose to call this latter property *local independence*.

Our paper was motivated by [8], where FBM's local independence property was needed, but attempts to find this result from literature were unsuccessful. Very recently, however, J. Picard [9] has proven the asymptotic orthogonality result using a different technique. The more functional analytic approach of the present note has the advantage of giving very precise estimates both for the rate of asymptotic orthogonality, and for the much stronger property of asymptotically vanishing mutual information.

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The structure of the paper is as follows: in the first section we briefly recall certain facts about Sobolev spaces with fractional smoothness index – these spaces are the main tool in our approach. We have tried to make the exposition readable for the readers with no previous knowledge on these spaces. The second section reviews the basic facts on the Gelfand-Yaglom theory of mutual information between Gaussian spaces. The third section contains the proof of our main results. The results are obtained in a quantitative form in terms of the relative size of the time intervals involved. Finally, the fourth section briefly considers the higher dimensional case and states open questions.

2. Preliminaries I: the fractional Sobolev spaces

We shall apply the common notation for uninteresting constants. They will all be denoted by the letter c, and its value can vary inside a single estimate. The notation $a \sim b$ means that the ratio of the (positive) quantities a and b stays bounded from below and above as the parameters of interest vary. The inner product of elements ϕ and ψ of a Hilbert space $\mathcal{H}$ will be denoted as $(\phi, \psi)_{\mathcal{H}}$, and the angle $\sphericalangle(A, B)$ between subspaces A and B of $\mathcal{H}$ is defined by

$$\cos(\sphericalangle(A,B)) := \sup\left\{ \frac{(U,V)_{\mathcal{H}}}{\|U\|_{\mathcal{H}}\|V_{\mathcal{H}}\|} : U \in A, V \in B \right\}.$$

Suitable references for this section are e.g. [10, Section 6] or selected parts of [12]. Vastly more information can be found in Triebel's monographs, like [15]. Actually only very little of the theory of Sobolev spaces is needed, and we try to be as self-contained as possible.

The Fourier transform of a tempered distribution f on $\mathbf{R}^n$ is defined as

$$\widehat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

We shall employ the notation $\langle \lambda, \mu \rangle$ for the distributional pairing, assuming that it is well-defined for λ and μ. Recall that the convolution $\lambda * \phi$ is always defined if λ is a Schwartz distribution and $\phi \in C_0^\infty(\mathbf{R}^n)$, and its Fourier transform is the product $(2\pi)^{n/2}\widehat{\phi}\,\widehat{\lambda}$. Moreover, by the definition of the Fourier transform, the Parseval identity can be written in the form

$$\langle \lambda, \overline{\phi} \rangle = \langle \widehat{\lambda}, \overline{\widehat{\phi}} \rangle.$$

Let $s \in \mathbf{R}$. The Sobolev space $W^{s,2}(\mathbf{R}^n)$ is defined as the Hilbert space of tempered distributions f on $\mathbf{R}^n$ such that the Fourier transform $\widehat{f}(\xi)$ is a locally integrable function with the property

$$\|f\|_{s,2} := \|f\|_{W^{s,2}} := \left(\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s \right)^{1/2} < \infty. \tag{1}$$

Our normalization constant for the Fourier transform makes sure that $W^{0,2}(\mathbf{R}^n) = L^2(\mathbf{R}^n)$ isometrically.

In the distributional pairing, the isometric dual of $W^{s,2}(\mathbf{R}^n)$ is $W^{-s,2}(\mathbf{R}^n)$. Moreover, the norm increases as s increases, and for integers $k \in \mathbf{N}$ we have that

$$\|f\|_{k,2}^2 \sim \int_{\mathbf{R}} (|f(x)|^2 + \sum_{|\alpha|=k} |f^{(\alpha)}(x)|^2)\, dx. \tag{2}$$

Obviously all these spaces are translation invariant, and one may verify that multiplication by an element in $C_0^\infty(\mathbf{R}^n)$ is continuous.

We next recall the homogeneous Sobolev spaces $\widetilde{W}^{s,2}(\mathbf{R}^n)$.The norm is replaced by

$$\|f\|_{\widetilde{W}^{s,2}(\mathbf{R}^n)} := \left(\int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2|\xi|^{2s}\, d\xi\right)^{1/2} < \infty. \tag{3}$$

This norm is certainly well-defined at least for all $f \in C_0^\infty(\mathbf{R}^n)$, although even then it may take the value ∞ if $s < -n/2$. In defining the Hilbert space $\widetilde{W}^{s,2}(\mathbf{R}^n)$ there indeed arises some complications in the definition, due to the fact that the $|\xi|^{2s}$ can be either 'too big' or 'too small' near origin. However, for our main result it is enough to consider the case $n = 1$ and $|s| < 1/2$, and then these difficulties disappear. For these values of the parameters the homogeneous spaces are simpy defined as the (inverse) Fourier transform of the weighted space $L^2_\mu(\mathbf{R})$, where the weight is of the form $\mu(d\xi) = |\xi|^{2s}$. By Cauchy-Schwartz any function in this weighted space is a locally integrable function, and thus defines a distribution in a natural way. On the other hand, every Schwartz test function belongs to this weighted space, which can be used to show that $C_0^\infty(\mathbf{R}) \subset \widetilde{W}^{s,2}$ is a dense subset. Moreover, the isometric duality $(\widetilde{W}^{s,2}(\mathbf{R}))^* = \widetilde{W}^{-s,2}(\mathbf{R})$ holds via the distributional duality

$$\langle \phi, \psi\rangle = \int_{\mathbf{R}} \phi(x)\psi(x)\, dx.$$

The pairing is originally defined only for test functions, but it extends to elements $\phi \in \widetilde{W}^{s,2}(\mathbf{R})$ and $\psi \in \widetilde{W}^{-s,2}(\mathbf{R})$ by continuity and density.

We then fix $s \in (-1/2, 1/2)$ together with an open interval $I \subset \mathbf{R}$ (I can well be unbounded) and define the Sobolev-functions over this interval. First of all we denote by $\widetilde{W}^{s,2}_0(I)$ the closure of $C_0^\infty(I)$ in the space $\widetilde{W}^{s,2}(\mathbf{R})$. Clearly all the elements in $\widetilde{W}^{s,2}_0(I)$ are distributions supported on $\overline{I}$. We will also need the space $\widetilde{W}^{s,2}(I)$ which consists of restrictions of elements of $\widetilde{W}^{s,2}(\mathbf{R})$ on the interval I. Thus $\widetilde{W}^{s,2}(I) = \{g_{|I} : g \in \widetilde{W}^{s,2}(\mathbf{R})\}$. This space is naturally normed by the induced quotient norm

$$\|f\|_{\widetilde{W}^{s,2}(I)} := \inf\{\|g\|_{\widetilde{W}^{s,2}(\mathbf{R})} : g_{|I} = f\}.$$

In a similar vain one defines the non-homogeneous space $W^{s,2}(I)$ by setting $W^{s,2}(I) = \{g_{|I} : g \in W^{s,2}(\mathbf{R})\}$ and introducing the quotient norm

$$\|f\|_{W^{s,2}(I)} := \inf\{\|g\|_{W^{s,2}(\mathbf{R})} : g_{|I} = f\}.$$

This definition makes sense for all $s \in \mathbf{R}$. One may easily verify that $\|f\|^2_{W^{1,2}(I)} \sim \int_I (f'^2(x) + f^2(x))\, dx$, where f' is the distributional derivative of f.

Since $\widetilde{W}^{s,2}_0(I) \subset \widetilde{W}^{s,2}(\mathbf{R})$ is a (closed) subspace, we deduce by standard Hilbert space theory that isometrically

$$(\widetilde{W}^{s,2}_0(I))' = \widetilde{W}^{-s,2}(I) \quad \text{and} \quad (\widetilde{W}^{s,2}(I))' = \widetilde{W}^{-s,2}_0(I) \tag{4}$$

through the pairing $\langle \phi, \psi\rangle = \int_I \phi(x)\psi(x)\, dx$ (extended again by continuity). There is thus a natural isometry $G : \widetilde{W}^{-s,2}(I) \to \widetilde{W}^{s,2}_0(I)$ in such a way that

$$(\phi, G\psi)_{\widetilde{W}^{s,2}_0(I)} = \int_I \phi(x)\psi(x)\, dx \tag{5}$$

for smooth elements ϕ and ψ. Again this extends for any $\phi \in \widetilde{W}^{s,2}_0(I)$ and $\psi \in \widetilde{W}^{-s,2}(I)$ by continuity.

In the Lemma below the assumption $|s| < \frac{1}{2}$ is crucial.

Lemma 1. *Let $s \in (-\frac{1}{2}, \frac{1}{2})$ and let $I \subset \mathbf{R}$ be an open interval of length 1.*

(i) *Multiplication by the signum function extends to a bounded linear operator on $\widetilde{W}^{s,2}(\mathbf{R})$. In other words, $\|\chi_{(-\infty,0)}f\|_{\widetilde{W}^{s,2}(\mathbf{R})}, \|\chi_{(0,\infty)}f\|_{\widetilde{W}^{s,2}(\mathbf{R})} \leq c\|f\|_{\widetilde{W}^{s,2}(\mathbf{R})}$ for all $f \in C_0^\infty(\mathbf{R})$. The same statement remains true if $\widetilde{W}^{s,2}(\mathbf{R})$ is replaced by $W^{s,2}(\mathbf{R})$.*

(ii) $\widetilde{W}_0^{s,2}(I) = \{f \in \widetilde{W}^{s,2}(\mathbf{R}) : \operatorname{supp}(f) \subset \overline{I}\}$.

(iii) *We have $\widetilde{W}_0^{s,2}(I) = \widetilde{W}^{s,2}(I) = W^{s,2}(I)$ with equivalent norms (the constant of isomorphism does not depend on the location of the interval I).*

(iv) *There is a continuous inclusion $W^{1,2}(I) \subset \widetilde{W}_0^{s,2}(I)$, and this natural imbedding is a Hilbert-Schmidt operator.*

Proof. (i) The statement is well-known, see [15, First Lemma in Section 2.10.2.]. Actually, up to a constant the multiplication by the signum function corresponds to the action of the Hilbert transfrom on the Fourier side. Hence the claim follows from the fact that $|\xi|^{2s}$ is a Muckenhoupt A^2-weight on $\mathbf{R}$ for any $s \in (-1/2, 1/2)$, see [13, Corollary, V.4.2, V.6.6.4]. In a similar way, by checking that $(1 + |\xi|^2)^s$ is a Muckenhoupt weight one obtains the statement concerning $W^{s,2}$.

(ii) Let $f \in \widetilde{W}^{s,2}(\mathbf{R})$ with $\operatorname{supp}(f) \subset \overline{I}$. We will show that one may approximate f in norm by the elements of $C_0^\infty(I)$. The dilation $\lambda \to f(\lambda\cdot)$ is a continuous map from a neighbourhood of 1 into $\widetilde{W}^{s,2}(\mathbf{R})$. Hence, by approximating f with a suitable dilation we may assume that $\operatorname{supp}(f)$ is contained in I. Finally, we then obtain the required approximant by a standard mollification.

(iii) By the translation invariance of the spaces, the independence on the location of the interval I is obvious. The first equality is an easy consequence of parts (i) and (ii). Towards the second equality, let us first verify that $C_0^\infty(I)$ is dense in $W^{s,2}(I)$. By part (i), if $f \in W^{s,2}(I)$ then also $\chi_I f \in W^{s,2}(\mathbf{R})$, where $\chi_I f$ stands for the zero continuation of f to $\mathbf{R}$. Exactly as in part (ii) we show by dilation and convolution approximation that $\chi_I f$ is in the closure of $C_0^\infty(I)$ in $W^{s,2}(\mathbf{R})$, which clearly yields the claim.

Hence it remains to show that

$$\|f\|_{W^{s,2}(\mathbf{R})} \sim \|f\|_{\widetilde{W}^{s,2}(\mathbf{R})} \quad \text{for } f \in C_0^\infty(I). \tag{6}$$

We may clearly assume that $I = (0, 1)$. Let us first consider the inequality

$$\|f\|_{W^{s,2}(\mathbf{R})} \leq c\|f\|_{\widetilde{W}^{s,2}(\mathbf{R})}. \tag{7}$$

This is immediate if $s \leq 0$. If $s \in (0, 1/2)$ we choose a cut-off function $\phi \in C_0^\infty(-1, 2)$ such that $\phi = 1$ on the interval $[-1/2, 3/2]$. Let us decompose

$$f = \phi f_1 + \phi f_2,$$

where $\widehat{f_1} = \chi_{[-1,1]}\widehat{f}$, and $\widehat{f_2} = \widehat{f} - \widehat{f_1}$. Then obviously $\|\phi f_2\|_{W^{s,2}(\mathbf{R})} \leq c\|f_2\|_{W^{s,2}(\mathbf{R})} \leq c\|f\|_{\widetilde{W}^{s,2}(\mathbf{R})}$. Moreover,

$$f_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ix\xi} \widehat{f}(\xi)\, d\xi, \quad f_1'(x) = \frac{i}{\sqrt{2\pi}} \int_{-1}^{1} e^{ix\xi} \xi \widehat{f}(\xi)\, d\xi \quad ,$$

where, by Cauchy-Schwarz, $\int_{-1}^{1} |\widehat{f}(\xi)| d\xi \leq c\|f\|_{\widetilde{W}^{s,2}(\mathbf{R})}$. Hence $\|f_1\|_\infty + \|f_1'\|_\infty \leq c\|f\|_{\widetilde{W}^{s,2}(\mathbf{R})}$ and we obtain that $\|\phi f_1\|_{W^{s,2}(\mathbf{R})} \leq \|\phi f_1\|_{W^{1,2}(\mathbf{R})} \leq c\|f_1\|_{\widetilde{W}^{s,2}(\mathbf{R})}$. By combining these estimates (7) follows.

In turn, the converse inequality

$$\|f\|_{\widetilde{W}^{s,2}(\mathbf{R})} \le c\|f\|_{W^{s,2}(\mathbf{R})}. \tag{8}$$

is immediate if $s \ge 0$. It clearly follows for negative $s \in (-1/2, 0)$ if we verify that in our situation $\|\widehat{f}\|_{L^\infty(-1,1)} \le c\|f\|_{W^{s,2}(\mathbf{R})}$. This is seen by observing that

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \langle f(x), \phi(x)e^{-i\xi x}\rangle,$$

where $\sup_{-1\le\xi\le 1} \|\phi(x)e^{-i\xi x}\|_{W^{-s,2}(\mathbf{R})} \le c$.

(iv) By part (iii), the claim is a consequence of the well-known Hilbert-Schmidt property of the inclusion $W^{1,2}(I) \subset W^{s,2}(I)$. Since we have not been able to find a convenient reference, the simple proof is sketched here. We may assume that $I = (-1/2, 1/2)$ so that $I \subset (-\pi, \pi] =: \mathbf{T}$, where $\mathbf{T}$ stands for the 1-dimensional torus. By applying a simple extension one may consider the spaces in question as closed subspaces of the corresponding Sobolev spaces $H^1(\mathbf{T})$ and $H^s(\mathbf{T})$ on the torus, where for $f = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ and $u \in \mathbf{R}$ one sets $\|f\|^2_{H^u(\mathbf{T})} = \sum_{n=-\infty}^{\infty}(1+|n|)^{2u}|a_n|^2$ (see e.g. [10]). By considering the natural orthogonal basis $((1+|n|)^{-u}e^{inx})_{n=-\infty}^{\infty}$ we see that the embedding $H^1(\mathbf{T}) \subset H^s(\mathbf{T})$ is equivalent to the diagonal operator with the diagonal elements $((1+|n|)^{s-1})_{n=-\infty}^{\infty}$. This is Hilbert-Schmidt as $\sum_{n=-\infty}^{\infty}(1+|n|)^{2s-2} < \infty$.

□

We shall need the formula for the Fourier transform of the function $u_\alpha(x) := |x|^{-\alpha}$, where $x \in \mathbf{R}^n$ and $\alpha \in (0, n)$. It is well-known, see e.g. [12, V 1. Lemma 2, p.117], that

$$\widehat{u}_\alpha(\xi) = d_{n,\alpha}|\xi|^{\alpha-n}, \quad \text{where } d_{n,\alpha} := 2^{n/2-\alpha}\frac{\Gamma((n-\alpha)/2)}{\Gamma(\alpha/2)}. \tag{9}$$

Lemma 2. *Assume that $s \in (-1/2, 1/2)$.*
(i) *Let $\alpha > 0$, $\alpha \ne 1$ and denote $f_\alpha(x) = (1+|x|)^{-\alpha}$. Then $f_\alpha \in \widetilde{W}^s(\mathbf{R})$ for $\alpha > 1/2 - s$.*
(ii) *Let $\alpha > 1/2 + s$, $\alpha \ne 1$. Then for any $k > 0$ there is a constant $c(\alpha, s) > 0$ such that $\|(k - \cdot)^{-\alpha}\|_{\widetilde{W}^{-s,2}((-\infty,0))} = c(\alpha, s)k^{1/2+s-\alpha}$. In other words,*

$$\sup_{\|\phi\|_{\widetilde{W}_0^{s,2}((-\infty,0))}\le 1} \int_{-\infty}^{0} (k-x)^{-\alpha}\phi(x)\,dx \; = c(\alpha, s)k^{1/2+s-\alpha}. \tag{10}$$

Proof. (i) Choose a smooth cut-off function $\phi \in C_0^\infty(\mathbf{R})$ such that $\phi = 1$ in a neighbourhood of the origin. Compose

$$f_\alpha(x) = \phi(x)f_\alpha(x) + (1-\phi(x))(f_\alpha(x) - |x|^{-\alpha}) + (1-\phi(x))|x|^{-\alpha} =: g_1(x) + g_2(x) + g_3(x).$$

Obviously $g_1 \in L^1(\mathbf{R}) \cap W^{1,2}(\mathbf{R}) \subset \widetilde{W}^{s,2}(\mathbf{R})$ for all $|s| < 1/2$. An easy eastimate shows that the same holds for g_2. Moreover, we observe that $(d/dx)g_3 \in L^2(\mathbf{R})$. Hence $\int_{\mathbf{R}} |\xi|^2|\widehat{g}_3(\xi)|^2 < 1$. Thus the inclusion $g_3 \in \widetilde{W}^{s,2}(\mathbf{R})$ holds if and only if the integral $\int_{-1}^{1} |\xi|^{2s}|\widehat{g}_3(\xi)|^2\,d\xi$ is finite.

Consider first the case $\alpha > 1$. Then $g_3 \in L^1(\mathbf{R})$, so that $\widehat{g}_3$ is bounded and $g_3 \in \widetilde{W}^{s,2}(\mathbf{R})$ for all $|s| < 1/2$. Assume then that $\alpha \in (0, 1)$. Then $g_3(x) - |x|^{-\alpha} \in L^1(\mathbf{R})$, so that (9) yields $|\widehat{g}_3(\xi) - d_{1,\alpha}|\xi|^{\alpha-1}| \le C$. Thus $\int_{-1}^{1} |\xi|^{2s}|\widehat{g}_3(\xi)|^2\,d\xi < \infty$ exactly for $s > 1/2 - \alpha$.

(ii) The definition of the homogeneous Sobolev norm yields the scaling rule $\|\phi(k\cdot)\|_{\widetilde{W}^{s,2}(\mathbf{R})} = k^{s-1/2}\|\phi(\cdot)\|_{\widetilde{W}^{s,2}(\mathbf{R})}$. By using this fact, duality, and a substitution $x = ky$ in the integral we are reduced to showing that

$$\sup_{\left\{\substack{\phi\in C_0^\infty(-\infty,0)\\ \|\phi\|_{\widetilde{W}^s(\mathbf{R})}\leq 1}\right.} \int_{-\infty}^0 (1+|x|)^{-\alpha}\phi(x)\,dx < \infty.$$

By duality this follows immediately from the fact that $(1+|x|)^{-\alpha} \in \widetilde{W}^{-s,2}(\mathbf{R})$ according to part (i) of the Lemma. $\square$

We finally remark that all the results stated in this section remain valid with identical proofs for the Sobolev spaces that contain only real-valued functions.

3. Preliminaries II: mutual information between Gaussian subspaces

In this section we present the needed facts from the Gelfand-Yaglom theory of mutual information between Gaussian subspaces. In order to recall the general concept of mutual information, let $(\Omega,\mathcal{F},P)$ be a probability space, and let $\mathcal{A}$ and $\mathcal{B}$ be sub-σ-algebras of $\mathcal{F}$. The mutual (Shannon) information between $\mathcal{A}$ and $\mathcal{B}$ is defined as [4]

$$I(\mathcal{A}:\mathcal{B}) := \sup_{\{A_j\}\{B_k\}} \sum_{k,j} \mathbf{P}(A_j\cap B_k)\log\left(\frac{\mathbf{P}(A_j\cap B_k)}{\mathbf{P}(A_j)\,\mathbf{P}(B_k)}\right).$$

Here the supremum is taken over all $\mathcal{A}$-measurable partitions $\Omega = \bigcup_{j=1}^n A_k$ and $\mathcal{B}$-measurable partitions $\Omega = \bigcup_{k=1}^m B_k$ of the probability space ($n,m\geq 1$, $\mathbf{P}(A_j)>0$ and $\mathbf{P}(B_k)>0$ for all j,k).

For random variables $X:\Omega\to E$ and $Y:\Omega\to F$, where E,F are measurable spaces, we set $I(X:Y) := I(\sigma(X):\sigma(Y))$. Let μ_X (resp. μ_Y, $\mu_{(X,Y)}$) be the distribution (measure) of X (resp. Y, (X,Y)) in the space E (resp. F, $E\times F$). Then, one may check that $I(X:Y)=\infty$ if the measure $\mu_{(X,Y)}$ is not absolutely continuous with respect to the product measure $\mu_X\otimes\mu_Y$. Moreover, in the case where $\mu_{(X,Y)} << \mu_X\otimes\mu_Y$ we denote $p = \frac{d\mu_{(X,Y)}}{d(\mu_X\otimes\mu_Y)}$ and have the formula

$$I(X:Y) = \int_{X\times Y} \log(p)\,d(\mu_X\otimes\mu_Y). \tag{11}$$

The Kullback-Leibler information characterizes the shift from a probability measure μ to another probability measure ν on the same measurable space, and it is defined as

$$I_{KL}(\mu:\nu) = \begin{cases} \int \log\frac{d\mu}{d\nu}d\nu, & \text{if } \mu << \nu,\\ \infty, & \text{otherwise.}\end{cases}$$

Shannon's mutual information can be expressed in terms of the Kullback-Leibler information as

$$I(\mathcal{A}:\mathcal{B}) = I_{KL}(P_{(\mathcal{A},\mathcal{B})} : P_{\mathcal{A}}\otimes P_{\mathcal{B}}), \tag{12}$$

where $P_{(\mathcal{A},\mathcal{B})}$ denotes the unique probability measure on $(\Omega\times\Omega, \mathcal{A}\times\mathcal{B})$ satisfying $P_{(\mathcal{A},\mathcal{B})}(A\times B) = P(A\cap B)$ for $A\in\mathcal{A}$, $B\in\mathcal{B}$. Actually, this is obtained from (11) by letting X (resp. Y) be the identity map $(\Omega,\mathcal{F})\to(\Omega,\mathcal{A})$ (resp. the identity map $(\Omega,\mathcal{F})\to(\Omega,\mathcal{B})$).

The following properties of mutual information are most conveniently proven through the relation (12).

Theorem 1. (i) *$I(\mathcal{A}:\mathcal{B})\geq 0$ and equality holds if and only if $\mathcal{A}$ and $\mathcal{B}$ are independent.*
(ii) *$I(\mathcal{A}:\mathcal{B})$ is non-decreasing with respect to $\mathcal{A}$ and $\mathcal{B}$.*
(iii) *If $\mathcal{A}_n\uparrow\mathcal{A}$ and $\mathcal{B}_n\uparrow\mathcal{B}$, then $I(\mathcal{A}_n:\mathcal{B}_n)\uparrow I(\mathcal{A}:\mathcal{B})$.*
(iv) *If $\mathcal{A}_n\downarrow\mathcal{A}$ and $\mathcal{B}_n\downarrow\mathcal{B}$, and if $I(\mathcal{A}_n:\mathcal{B}_n)<\infty$ for some n, then $I(\mathcal{A}_n:\mathcal{B}_n)\downarrow I(\mathcal{A}:\mathcal{B})$.*

When X and Y are finite-dimensional random vectors such that (X,Y) is a non-degenerate and centered multivariate Gaussian, one may easily compute by using (11) that

$$I(X:Y)=\frac{1}{2}\log\frac{\det(\Gamma_X)\det(\Gamma_Y)}{\det(\Gamma_{(X,Y)})},$$

where Γ_Z denotes the covariance matrix of a Gaussian vector Z. In particular, the information between random variables X and Y with bivariate centered Gaussian distribution is

$$I(\sigma(X):\sigma(Y))=-\log\sin\sphericalangle(X,Y). \tag{13}$$

The theory of Shannon information between Gaussian processes was developed by Gel'fand and Yaglom [5]. Their fundamental discovery was that one may express the information between two closed subspaces A and B of a Gaussian space $\mathcal{G}$ in terms of the spectral properties of the operator $T:=P_AP_BP_A$, where P_A and P_B stand for the orthogonal projections on A and B, respectively. In order to explain their result, and for later purposes, we first recall some basic notions of operator theory.

Let $S:E\to F$ be a bounded linear operator between the separable Hilbert spaces E and F. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for E. The Hilbert-Schmidt norm of S is defined as

$$\|S\|_{HS(E,F)}:=\big(\sum_{i\in I}\|Se_i\|_F^2\big)^{1/2}.$$

This definition does not depend on the particular orthonormal basis used. In case $\|S\|_{HS}<\infty$ we say that S is a Hilbert-Schmidt operator. Also it is clear that if E (resp. F) is a Hilbert subspace of a larger space $\widetilde{E}$ (resp. $\widetilde{F}$), then $\|SP_E\|_{HS(\widetilde{E},\widetilde{F})}=\|S\|_{HS(E,F)}$. In this sense it is not important to keep exact track on the domain of definition and image spaces, and one usually abbreviates $\|S\|_{HS(E,F)}=\|S\|_{HS}$. For products of bounded linear operators between (perhaps different) Hilbert spaces we have

$$\|TS\|_{HS}\leq\|T\|_{HS}\|S\|\quad\text{and}\quad\|ST\|_{HS}\leq\|S\|\|T\|_{HS}. \tag{14}$$

Let us then assume, in addition, that $S:E\to E$ is self-adjoint and positive semi-definite, $S^*=S$ and $S\geq 0$. Then one may always define the trace of S by setting

$$\mathrm{tr}\,(S):=\sum_{i\in I}(e_i,Se_i)$$

Thus, $\mathrm{tr}\,(S)\in[0,\infty]$. In the case that $\mathrm{tr}\,(S)<\infty$ we say that S is of trace class. Every trace class operator S is compact, and since we also assume $S\geq 0$, it has a decreasing sequence of positive eigenvalues $\lambda_1\geq\lambda_2\geq\ldots\geq 0$, where each eigenvalue is counted according to its multiplicity. It follows that

$$\mathrm{tr}\,(S)=\sum_{\lambda_k>0}\lambda_k. \tag{15}$$

We finally observe that if $S : E \to F$ is any bounded linear operator, then $S^*S \geq 0$ is self-adjoint, and we may compute

$$\mathrm{tr}\,(S^*S) = \|S\|_{HS}^2. \tag{16}$$

Let us then go back to the situation where A, B are closed subspaces of a Gaussian Hilbert space $\mathcal{G}$ and state the result of Gelfand and Yaglom. Again P_A and P_B stand for the orthogonal projections to the subspaces A and B, respectively, and $I(A : B) := I(\sigma\{X : X \in A\} : \sigma\{Y : Y \in B\})$.

Theorem 2. [5] *Denote $T := P_A P_B P_A$. The mutual information $I(A : B)$ is finite if and only if $\|T\| < 1$ (i.e. $\sphericalangle(A, B) > 0$) and the operator T is of trace class. Moreover, in this case*

$$I(A : B) = \frac{1}{2} \sum_{k:\lambda_k>0} \log(\frac{1}{1-\lambda_k}), \tag{17}$$

where $\lambda_1 \geq \lambda_2 \geq \ldots$ are the eigenvalues of T in the decreasing order repeated according to their multiplicities.

A nice sketch of the derivation of the formula (17) is included in a form of exercises in [3, pp. 68–69]. Assume that T is of trace class, and let $Z_1, Z_2, \ldots$ be an orthonormal basis of $T\mathcal{G}$ consisting of eigenvectors corresponding to the non-zero eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$. It is not difficult to see that $\{P_B Z_i\}$ is an orthogonal basis of $P_B P_A P_B \mathcal{G}$, and, moreover, these bases are *mutually* orthogonal: $(Z_i, P_B Z_j) = 0$ for $i \neq j$. Since orthogonality implies independence in the case of Gaussian random variables, it follows that the information between $\sigma(A)$ and $\sigma(B)$ can be expressed as the sum of the informations within the pairs $(Z_i, P_B Z_i)$, given in (13):

$$\begin{aligned} I(A : B) &= -\sum_i \log\sin \sphericalangle(Z_i, P_B Z_i) \\ &= -\frac{1}{2}\sum_i \log(1-\cos^2 \sphericalangle(Z_i, P_B Z_i)) = -\frac{1}{2}\sum_i \log(1-\lambda_i). \end{aligned}$$

Note that since $\sphericalangle(A, B) = \inf_i \sphericalangle(Z_i, P_B Z_i)$, the information between subspaces can be infinite even when they have a positive angle.

By invoking the Taylor series of $x \mapsto \log(1/(1-x))$ we obtain for $x \in [0, 1)$ that

$$x \leq \log(\frac{1}{1-x}) = \sum_{k=1}^{\infty} \frac{1}{k} x^k \leq x + \frac{1}{2} x^2 (\frac{1}{1-x}) \leq x(1 + \frac{x}{2(1-x)}). \tag{18}$$

Observe also that $T = (P_B P_A)^*(P_B P_A)$ and $\|T\| = \|P_B P_A\|^2$. Moreover, $\lambda_1 \leq \|P_B P_A\| \leq \|P_B P_A\|_{HS}$. By combining these observations and the facts (15)–(18) we obtain a formulation suitable for our purposes:

Corollary 1. *The angle between the spaces A and B satisfies $\cos(\sphericalangle(A, B)) = \|P_B P_A\|$. We have $I(A : B) < \infty$ if and only if $\|P_B P_A\| < 1$ and $\|P_B P_A\|_{HS} < \infty$. Moreover, in this case*

$$\frac{1}{2}\|P_B P_A\|_{HS}^2 \quad \leq \quad I(A : B) \;\leq\; \frac{1}{2}\|P_B P_A\|_{HS}^2 \left(1 + \frac{\|P_B P_A\|}{2(1 - \|P_B P_A\|)}\right) \tag{19}$$

Observe that the above estimate is asymptotically precise in the limit $\|P_B P_A\| \to 0$, or, equivalently, as $\sphericalangle(A, B) \to \pi/2$. Especially this is true in the limit $I(A : B) \to 0$.

4. Statement and proof of the main results

In this section we consider the asymptotic independence of the local spaces of FBMs. To be more exact, let us first define for any set $S \subset \mathbf{R}$

$$E_S := \overline{\operatorname{span}}\{X_u - X_v : u, v \in S\},$$

and the shorthand notation

$$E_{t,\varepsilon} := E_{(t-\varepsilon,t+\varepsilon)}.$$

We consider the following two notions of local independence.

Definition 1. *We say that the stochastic process X possesses* local independence in the weak sense, *if for any distinct t_1, t_2*

$$\sphericalangle(E_{t_1,\varepsilon}, E_{t_2,\varepsilon}) \to \frac{\pi}{2} \quad \text{as } \varepsilon \searrow 0.$$

We say that the stochastic process X possesses local independence (in the strong sense), *if for any distinct t_1, t_2*

$$I(E_{t_1,\varepsilon} : E_{t_2,\varepsilon}) \to 0 \quad \text{as } \varepsilon \searrow 0.$$

The term 'weak' corresponds to its use in 'stationarity in the weak sense'.

We will consider integrals of the form $\int_{\mathbf{R}} X_t\phi(t)dt$ for smooth and compactly supported functions ϕ. The definition of the integral poses no problems since $t \mapsto X_t$ is continuous with respect to L^2-norm of random variables, whence it can be e.g. defined as the limit of the corresponding Riemann sums (or as a Bochner integral). Let us start with two simple lemmata.

Lemma 3. *For any $T \in \mathbf{R}$ and $a > 0$ the elements*

$$\int_{\mathbf{R}} \phi'(t)X_t\, dt, \quad \phi \in C_0^\infty(T, T+a) \tag{20}$$

are dense in $E_{(T,T+a)}$.

Proof. By observing that $\int_{\mathbf{R}} \phi'(t)X_t\, dt = \int_{\mathbf{R}} \phi'(t)(X_t - X_{T+a/2})\, dt$ we see that the elements in question are contained in $E_{T,a}$. Conversely, let $\phi \in C_0^\infty(\mathbf{R})$ satisfy $\int_{\mathbf{R}} \phi(t) = 1$. Denote $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x\varepsilon)$. By the L^2-continuity we have that for any $t_1, t_2 \in (T, T+a)$

$$X_{t_1} - X_{t_2} = \lim_{\varepsilon\to 0}(\int_{\mathbf{R}} X_u(\phi_\varepsilon(t_1+u) - \phi_\varepsilon(t_2+u))\, du.$$

Observe that we may write $\psi' = \phi_\varepsilon(t_1+\cdot) - \phi_\varepsilon(t_2+\cdot)$ for suitable $\psi \in C_0^\infty(\mathbf{R})$. This yields the claim. $\square$

Next we verify that the L^2-norm of a random variable of the form (20) equals the norm of ϕ in a corresponding homogeneous Sobolev space. For later purposes we first state an auxiliary result that is valid in all dimensions.

Lemma 4. *Assume that $H \in (0,1)$ and the functions $\phi, \psi \in C_0^\infty(\mathbf{R}^n)$ satisfy $\int_{\mathbf{R}^n} \phi\, dx = \int_{\mathbf{R}^n} \psi\, dx = 0$. Then*

$$\begin{aligned} &\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} \frac{1}{2}\big(|u|^{2H} + |v|^{2H} - |u-v|^{2H}\big)\phi(u)\overline{\psi}(v)\, dudv \\ &= -2^{n+2H-1}\pi^{n/2}\frac{\Gamma(n/2+H)}{\Gamma(-H)})(\phi,\psi)_{\widetilde{W}^{-n/2-H,2}(\mathbf{R}^n)}. \end{aligned} \tag{21}$$

.

Proof. We first claim that for $\alpha \in (0,n)$

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |u-v|^{-\alpha}\phi(u)\overline{\psi(v)}\,dudv = (2\pi)^{n/2} d_{n,\alpha}\int_{\mathbf{R}^n} |\xi|^{\alpha-n}\widehat{\phi}(\xi)\overline{\widehat{\phi}(\xi)}\,d\xi. \tag{22}$$

This is immediate by (9) and the Parseval formula since the left hand side above can be written as $\int_{\mathbf{R}^n} g\overline{\psi}\,dx$ where g is obtained as the convolution $g = u_\alpha * \phi$, whence its Fourier transform equals $(2\pi)^{n/2} d_{n,\alpha}|\xi|^{n-\alpha}\widehat{\phi}(\xi)$. By the assumption we see that the Fourier transforms of ϕ and ψ satisfy $|\phi(\xi)|, |\psi(\xi)| \leq c|\xi|$ near the origin. Moreover, they decay polynomially as $|\xi| \to \infty$. These observations verify that the right hand side of (22) is analytic as a function of α in a neighbourhood of the open line segment $\alpha \in (-2,n)$. Since the left hand side of (22) is likewise analytic in the same neighbourhood we deduce by analytic continuation that (22) holds true for all $\alpha \in (-2,n)$. The claim follows as we substitute $\alpha = -2H$ in (22) and observe that by Fubini the terms $|u|^{2H}$ and $|v|^{2H}$ make no contribution to the integral in the left hand side of (21). □

Corollary 2. *Let $H \in (0,1)$ and assume that $\phi_1, \phi_2 \in C_0^\infty(\mathbf{R})$ are real-valued. Then*

$$\mathbf{E}\left((\int_{\mathbf{R}} \phi_1'(t)X_t\,dt)(\int_{\mathbf{R}} \phi_2'(t)X_t\,dt)\right) = a_H(\phi_1,\phi_2)_{\widetilde{W}^{\frac12 - H,2}},$$

where $a_H := \sin(\pi H)\Gamma(1+2H) > 0$. Especially, there is an isometric and bijective isomorphism $J : E_{(-\infty,\infty)} \to \widetilde{W}^{1/2-H,2}(\mathbf{R})$ so that for each interval $(t,t') \subset \mathbf{R}$ we have $J(E_{(t,t')}) = \widetilde{W}_0^{1/2-H,2}((t,t'))$.

Proof. Let us denote

$$A := \mathbf{E}\left((\int_{\mathbf{R}} \phi_1'(t)X_t\,dt)(\int_{\mathbf{R}} \phi_2'(t)X_t\,dt)\right)$$

By the definition of the fractional Brownian motion with the Hurst parameter $H \in (0,1)$ we have

$$A = \frac{1}{2}\int_{\mathbf{R}\times\mathbf{R}} \phi_1'(u)_2\phi'(s)(|s|^{2H} + |u|^{2H} - |s-u|^{2H})\,ds\,du \tag{23}$$

$$= a_H(\phi_1', \phi_2')_{\widetilde{W}^{-1/2-H,2}} = a_H(\phi_1,\phi_2)_{\widetilde{W}^{\frac12 - H,2}}. \tag{24}$$

Above we used Lemma 4 to obtain the first equality. Observe that the functions ϕ_1' and ϕ_2' automatically have mean zero. The last equality follows directly from the fact that the Fourier transfrom of ϕ_j' equals $i\xi\widehat{\phi}_j(\xi)$, $j = 1,2$. The constant is simplified by applying the standard formulas for the Gamma functions, see e.g. [1, 5.2.4]. The last statement of the Corollary follows immediately by Lemma 3. □

Remark 3. Note that a_H takes the value 1 for $H = 1/2$ and tends to zero as $H \to 1^-$ or $H \to 0^+$.

Let us observe that if the supports of ϕ_1 and ϕ_2 are disjoint, we are free to integrate by parts in (23) and obtain the formula

$$\mathbf{E}\left((\int_{\mathbf{R}} \phi_1'(t)X_t\,dt)(\int_{\mathbf{R}} \phi_2'(t)X_t\,dt)\right) = H(2H-1)\int_{\mathbf{R}\times\mathbf{R}} \frac{\phi_1(u)\phi_2(v)}{|u-v|^{2-2H}}\,dudv. \tag{25}$$

Here it is interesting to observe the sign of the factor $H(2H-1)$ for different values of the Hurst parameter H.

We are now ready to prove the main result of the paper.

Theorem 4. *Fractional Brownian motions with $H \in (0,1)$ possess local independence. Moreover, there is a constant $r_H \geq 0$ (with $r_H > 0$ for $H \neq 1/2$) such that*

$$\cos(\sphericalangle(E_{t_1,\varepsilon}, E_{t_2,\varepsilon}) \;=\; r_H(\varepsilon/|t_1-t_2|)^{2-2H} + O(\varepsilon^{3-2H}) \quad \text{as } \varepsilon \to 0,$$

and (with some $\delta_H > 0$)

$$I(E_{t_1,\varepsilon} : E_{t_2,\varepsilon}) \;=\; \frac{1}{2} r_H^2(\varepsilon/|t_1-t_2|)^{4-4H} + O(\varepsilon^{4-4H+\delta_H}) \quad \text{as } \varepsilon \to 0.$$

Proof. By scaling invariance and stationarity it is equivalent to show that

$$\cos(\sphericalangle(E_{(0,1)}, E_{(k,k+1)})) \;=\; r_H k^{2H-2} + O(k^{2H-3}) \quad \text{as } k \to \infty \quad \text{and} \tag{26}$$

$$I(E_{(0,1)} : E_{(k,k+1)}) \;=\; \frac{1}{2} r_H^2 k^{4H-4} + O(k^{4H-5}) \quad \text{as } k \to \infty. \tag{27}$$

Denote $s := 1/2 - H \in (-1/2, 1/2)$ together with $A := \widetilde{W}_0^{s,2}(k,k+1)$ and $B := \widetilde{W}_0^{s,2}(0,1)$, considered as subspaces of the Hilbert space $\widetilde{W}^{s,2}(\mathbf{R})$. Let P_A (resp. P_B) stand for the orthogonal projection on A (resp. B). We will consider the operator

$$S := P_B : A \to B.$$

Since $S = (P_B P_A)_{|A}$ and $(P_B P_A)_{|A^\perp} = 0$, we obtain that $\|S\| = \|P_B P_A\|$ and $\|S\|_{HS} = \|P_B P_A\|_{HS}$. Hence Corollaries 1 and 2 yield that

$$\cos(\sphericalangle(E_{(0,1)}, E_{(k,k+1)})) = \|S\| \tag{28}$$

and

$$\frac{1}{2}\|S\|_{HS}^2 \leq I(E_{(0,1)}, E_{(k,k+1)}) \leq \frac{1}{2}\|S\|_{HS}^2(1 + \|S\|) \tag{29}$$

as soon as $\|S\| < 1/2$.

In order to estimate the norm and the Hilbert-Schmidt norm of the operator S we will make use of the decay of the kernel in (25), and the even faster decay of its derivatives. For that end we need to first factorize S properly through a suitable integral operator. Assume thus that $k \geq 2$ and $\phi \in C_0^\infty(k,k+1) \subset A$. Then by definitions and formula (25) we see that $S\phi \in B$ is the unique element that satisfies for each $\psi \in C_0^\infty(0,1)$

$$\begin{aligned} (S\phi, \psi)_{\widetilde{W}_0^{s,2}(0,1)} &= (\phi, \psi)_{\widetilde{W}_0^{s,2}(\mathbf{R})} = H(2H-1) \int_{(0,1)\times(k,k+1)} \frac{\phi(y)\psi(x)}{|x-y|^{2-2H}}\, dxdy \\ &= \int_0^1 \psi(x)(R\phi)(x)\, dx, \end{aligned} \tag{30}$$

where R stands for the integral operator

$$R\phi(x) := H(2H-1) \int_{(k,k+1)} \frac{\phi(y)}{|x-y|^{2-2H}}\, dy.$$

By the smoothness of the kernel we immediately see that R is well-defined and, in fact

$$R\,(\widetilde{W}_0^{s,2}(k,k+1)) \subset W^{1,2}((0,1)).$$

Let $G : \widetilde{W}^{-s,2}(0,1) \to \widetilde{W}_0^{s,2}((0,1))$ be the isometric isomorphism from (5). According to (30) we may factorize

$$S = GR.$$

Let $V : \widetilde{W}_0^{s,2}(k,k+1) \to \widetilde{W}^{-s,2}(0,1)$ stand for the one-dimensional operator

$$V\phi(x) := \int_{(k,k+1)} \phi(y)\,dy, \quad \text{for } x \in (0,1).$$

Thus $V\phi$ is constant on $(0,1)$. We decompose

$$S \;=\; H(2H-1)k^{2H-2}GV + G\big(R - H(2H-1)k^{2H-2}V\big).$$

If we show that

$$\| \Big(R - H(2H-1)k^{2H-2}V\Big) : \widetilde{W}_0^{s,2}(k,k+1) \to \widetilde{W}^{-s,2}(k,k+1)\|_{HS} \tag{31}$$
$$= \; O(k^{2H-3}),$$

then, according to (28)-(29) and the fact that for the one-dimensional operator GV it holds that $\|GV\|_{HS} = \|GV\|$ (the value is independent of k), both of the asymptotics in (26) follow immediately. Here we also keep in mind that the Hilbert-Schmidt norm always dominates the operator norm.

Observe towards (31) that for $x \in (0,1)$ and $\phi \in C_0^\infty(k,k+1)$ we may write

$$\big((R - (H(2H-1)k^{2H-2}V)\phi\big)(x) = c\int_{(k,k+1)} u(x,y)\phi(y)\,dy,$$

where a simple computation shows that the the kernel $u(x,y) = |x-y|^{2H-2} - k^{2H-2}$ satisfies

$$\|\big((\frac{d}{dx})^\alpha(\frac{d}{dy})^\beta u\big)(x,\cdot)\|_{L^\infty(k,k+1)} \le ck^{2H-3}, \quad \alpha,\beta \in \{0,1\},\ x \in (0,1).$$

By Lemma 1(iii) we have $\|\cdot\|_{\widetilde{W}^{-s,2}((k,k+1))} \le \|\cdot\|_{W^{1,2}((k,k+1))}$. Hence the previous estimates yield for fixed $x \in (0,1)$ the estimate

$$\|u(x,\cdot)\|_{\widetilde{W}^{-s,2}((k,k+1))} \le \|u(x,\cdot)\|_{W^{1,2}((k,k+1))} \le c'k^{2H-3} \tag{32}$$

and, similarly

$$\|(\frac{d}{dx})u(x,\cdot)\|_{\widetilde{W}^{-s,2}((k,k+1))} \le \|(\frac{d}{dx})u(x,\cdot)\|_{W^{1,2}((k,k+1))} \le c'k^{2H-3}. \tag{33}$$

Assume that $\|\phi\|_{\widetilde{W}_0^{s,2}((k,k+1))} = 1$. The duality (4), estimates (32) and (33) show that

$$\max_{\alpha\in\{0,1\}} \|(\frac{d}{dx})^\alpha\Big(\big(R - (H(2H-1)k^{2H-2}V\big)\phi\Big)\|_{L^\infty(0,1)} \le c'k^{2H-3}.$$

This especially implies that

$$\|\Big(R - (H(2H-1)k^{2H-2}V\Big) : \widetilde{W}_0^{s,2}(k,k+1) \to W^{1,2}(k,k+1)\| \le c_2k^{2H-3}.$$

Let us denote by $I : W^{1,2}((0,1)) \to \dot{W}^{-s,2}((0,1))$ the natural imbedding. According to Lemma 1 (iv) we have $\|I\|_{HS} < \infty$. We finally obtain

$$\begin{aligned} &\|\big(R - (H(2H-1)k^{2H-2}V\big) : \widetilde{W}_0^{s,2}(k,k+1) \to \widetilde{W}^{-s,2}(k,k+1)\|_{HS} \\ \le\; &\|I\|_{HS}\|\big(R - (H(2H-1)k^{2H-2}V\big) : \widetilde{W}_0^{s,2}(k,k+1) \to W^{1,2}(k,k+1)\| \\ \le\; &c_3k^{2H-3}. \end{aligned}$$

This establishes (31) and completes the proof of the theorem. □

Remark 5. A closer inspection of the above proof reveals that the constant r_H in Theorem 4 satisfies $r_H = H|2H-1|\|\chi_{(0,1)}\|^2_{\widetilde{W}^{H-1/2,2}(0,1)}$. Especially, r_H tends to zero as $H \to 1/2$. Moreover, one also checks that it is possible to choose $\delta_H = \min(1, 2-2H)$.

After Theorem 4 it is natural to ask whether similar phenomena take place if only one of the intervals in consideration tends to a point. The answer is positive again. Heuristically one might expect that the speed of convergence is only half of what it was before, and this actually turns out to be true.

Theorem 6. *Let $t>0$. Then there are constants $r'_H \geq 0$ (with $r'_H>0$ for $H\neq 1/2$) and $\delta'_H>0$ such that as $\varepsilon\to 0$ one has*

$$\begin{aligned}\cos(\sphericalangle(E_{(-\infty,0)},E_{t,\varepsilon})) &= r'_H(\varepsilon/t)^{1-H}+O(\varepsilon^{2-H}) \qquad \textit{and}\\ I(E_{(-\infty,0)}:E_{t,\varepsilon}) &= \frac{1}{2}(r'_H)^2(\varepsilon/t)^{2-2H}+O(\varepsilon^{2-2H+\delta'_H}) \quad \textit{as } \varepsilon\to 0.\end{aligned}$$

Proof. As in the proof of Theorem 4 we apply scaling, Corollaries 1 and 2, and Lemma 1(iv) to the effect that it is equivalent to verify in the limit $k\to\infty$ that we have

$$\|\dot S\| = r'_H k^{H-1}+O(k^{H-2}) \quad\text{and}\quad \|\widetilde S\|_{HS} = r'_H k^{H-1}+O(k^{H-2}). \tag{34}$$

Here $\dot S=\dot G\dot R$, where $\dot G$ stands for the natural isomorphism $\dot G:\widetilde W^{-s,2}(k,k+1)\to \widetilde W_0^{s,2}((k,k+1))$ provided by (5), $s:=1/2-H$, and

$$\dot R:\widetilde W_0^{s,2}((-\infty,0))\to \widetilde W^{-s,2}(k,k+1)$$

is the integral operator

$$\dot R\phi(x):=H(2H-1)\int_{-\infty}^0 \frac{\phi(y)}{|x-y|^{2-2H}}\,dy,\quad \text{for } x\in(k,k+1).$$

This time we consider the auxiliary operator $\dot V:\widetilde W_0^{s,2}((-\infty,0))\to\widetilde W^{-s,2}(k,k+1)$, where

$$\dot V\phi(x):=H(2H-1)\int_{-\infty}^0 \frac{\phi(y)}{|k-y|^{2-2H}}\,dy,\quad \text{for } x\in(k,k+1).$$

Thus $\dot V$ is one-dimensional since its image contains only constant functions.

According to Lemma 2 it holds that

$$\|\,|k-\cdot|^{2H-2}\|_{\widetilde W^{-s,2}((-\infty,0))}=ck^{H-1}.$$

Hence, by one-dimensionality and the duality (4) we infer that

$$\|\dot V:\widetilde W_0^{s,2}((-\infty,0))\to\widetilde W^{-s,2}(k,k+1)\|=c'k^{H-1}.$$

By using again the decomposition $\dot S=\dot G\dot V+\dot G(\dot R-\dot V)$ we deduce, as in the proof of Theorem 4, that the one-dimensionality of $\dot V$ and the Hilbert-Schmidt property of the natural imbedding $W^{1,2}((k,k+1))\to\widetilde W^{-s,2}((k,k+1))$ (where the Hilbert-Schmidt norm is independent of k) enable us to deduce (34) as soon as we establish that

$$\|\big(\dot R-\dot V\big):\widetilde W_0^{s,2}(-\infty,0)\to W^{1,2}(k,k+1)\|\leq c_2k^{H-2}. \tag{35}$$

Observe that $\dot V-\dot R$ has the integral kernel $\dot u(x,y):=2(2H-1)\big(|x-y|^{2H-2}-|k-y|^{2H-2}\big)$. Clearly (35) follows from duality and the estimate

$$\sup_{x\in(k,k+1)}\|(\frac{d}{dx})^\alpha \dot u(x,\cdot)\|_{\widetilde W^{-s,2}((-\infty,0))}=O(k^{H-2})\quad \text{for } \alpha\in\{0,1\}. \tag{36}$$

In turn, for $\alpha = 1$ this estimate is a direct consequence of Lemma 3. In order to verify it for $\alpha = 0$, we fix $x \in (k, k+1)$ and apply the same Lemma as follows:

$$\begin{aligned} \|\dot{u}(x,\cdot)\|_{\widetilde{W}^{-s,2}((-\infty,0))} &= c\| \int_k^x |t-\cdot|^{2H-3}\, dt \|_{\widetilde{W}^{-s,2}((-\infty,0))} \\ &\leq c\int_k^x \|\, |t-\cdot|^{2H-3} \|_{\widetilde{W}^{-s,2}((-\infty,0))} dt \leq c' k^{H-2}. \end{aligned}$$

In the second inequality above we made use of the Minkowski inequality for Banach space norms. □

The remaining cases are simpler to handle and they are collected in the following theorem.

Theorem 7. (i) *Let $H \neq \frac{1}{2}$. Then $I(E_{(-\varepsilon,0)} : E_{(0,\varepsilon)}) = \infty$ for any $\varepsilon > 0$.*
(ii) *Let $H \neq \frac{1}{2}$. Then $I(E_{(-\infty,-\varepsilon)} : E_{(\varepsilon,\infty)}) = \infty$ for any $\varepsilon > 0$.*
(iii) $\sphericalangle(E_{(-\infty,0)}, E_{(0,\infty)}) > 0$.
(iv) *Let $t_1 < t < t_2$ be arbitrary. Then for small enough $\varepsilon > 0$ it holds that*

$$I(E_{(-\infty,t_1)\cup(t_2,\infty)} : E_{t,\varepsilon}) \leq c\varepsilon^{H-1}. \tag{37}$$

Proof. (i) Assume the contrary, that is, $I(E_{(-\varepsilon,0)} : E_{(0,\varepsilon)}) < \infty$ for some $\varepsilon > 0$. Since FBM possesses local independence, its infinitesimal space is trivial, that is, $\bigcap_{n=1}^{\infty} E_{(0,\pm\varepsilon/n)} = \{0\}$ (otherwise the Gaussian space would have uncountable dimension; see Proposition 5 of [8]). By Theorem 1 of [16], this implies the corresponding relation for σ-algebras, i.e. $\bigcap_{n=1}^{\infty} \sigma(E_{(0,\pm\varepsilon/n)}) = \{\Omega, \emptyset\}$ up to sets of measure 0 or 1. Theorem 1 (iv) then yields that $\lim_{n\to\infty} I(E_{(-\varepsilon/n,0)} : E_{(0,\varepsilon/n)}) = I(\{\Omega,\emptyset\} : \{\Omega,\emptyset\}) = 0$. On the other hand, we have $I(E_{(-\varepsilon,0)} : E_{(0,\varepsilon)}) > 0$ when $H \neq \frac{1}{2}$. Now, however, the self-similarity of FBM implies that $I(E_{(-\varepsilon/n,0)} : E_{(0,\varepsilon/n)})$ does not depend on n, and we get a contradiction.

(ii) By self-similarity, Theorem 1 (iii) and the previous claim, we have

$$\begin{aligned} I(E_{(-\infty,-\varepsilon)} : E_{(\varepsilon,\infty)}) &= \lim_{n\to\infty} I(E_{(-\infty,-\varepsilon/n)} : E_{(\varepsilon/n,\infty)}) \\ &= I(E_{(-\infty,0)} : E_{(0,\infty)}) \\ &\geq I(E_{(-\varepsilon,0)} : E_{(0,\varepsilon)}) = \infty. \end{aligned}$$

(iii) This is an immediate consequence of Lemma 1(i) and Lemma 3, since together they imply that for a dense set of elements $X_1 \in E_{(-\infty,0)}$ and $X_2 \in E_{(0,\infty)}$ we have that

$$\max(\|X_1\|, \|X_2\|) \leq c\|X_1 - X_2\|.$$

(iv) Write $A_1 = E_{(-\infty,t_1)}$, $A_2 = E_{(t_2,-\infty)}$, and $B = E_{t,\varepsilon}$. Since the angle between the subspaces A_1 and A_2 is positive, we see that $A := \overline{\mathrm{span}}(A_1 \bigcup A_2)$ is naturally isomorphic (not necessarily isometric) to the direct sum $(A_1 \oplus A_2)_{\ell^2}$. In this isomorphism the operator $P_B : A \to B$ conjugates to the operator $[P_B : A_1 \to B,\ P_B : A_2 \to B]$, whose Hilbert-Schmidt norm is bounded by $c\varepsilon^{1-H}$ by Theorem 6. This proves the claim. □

5. Generalizations and open questions

The most natural generalization of FBM to $\mathbf{R}^n$ is the Levy FBM, which is defined as the Gaussian process X_u indexed by the parameter $u \in \mathbf{R}^n$ and having the

covariance structure

$$\mathbf{E}\, X_u X_v = \frac{1}{2}\big(|u|^{2H} + |v|^{2H} - |u-v|^{2H}\big).$$

Here $H \in (0,1)$. As in the one-dimensional case this process has a version that has Hölder continuous realizations. We refer to [6, Chapter 18] for the existence and basic properties of n-dimensional Levy FBM. We will sketch the proof of an n-dimensional version of Theorem 4. For that end we first present an auxiliary result.

Lemma 5. *Let $n \geq 2$ and $s \in (-n/2-1, -n/2)$. Then there is a constant $c > 0$ such that for every $\phi \in C_0^\infty(B(0,1))$ with $\int_{\mathbf{R}^n} \phi\, dx = 0$ and $f \in C^{n+1}(\overline{B(0,1)})$ it holds that*

$$|\int_{B(0,1)} f\phi\, dx| \leq c\|\phi\|_{\widetilde{W}_0^{s,2}(B(0,1))} \sum_{1\leq|\alpha|\leq n+1} \|D^\alpha f\|_{L^\infty(\overline{B(0,1)})}.$$

Proof. Observe that in the left hand side we may replace f by $f-m$, where m is the average of f over the ball $B(0,1)$. Hence we may assume that $\|f\|_{L^\infty(\overline{B(0,1)})}$ is dominated by $\|Df\|_{L^\infty(\overline{B(0,1)})}$. It follows that it is enough to prove the stated estimate where one sums over all $|\alpha| \leq n+1$ in the right hand side. But it is easy to extend f to an element $\widetilde{f} \in W^{n+1,2}(\mathbf{R}^n)$ with norm less than constant times $\sum_{|\alpha|\leq n+1} \|D^\alpha f\|_{L^\infty(\overline{B(0,1)})}$. The claim follows now by duality since formally $W^{n+1,2}(B(0,1)) \subset \widetilde{W}^{-s,2}(B(0,1)) = \widetilde{W}_0^{s,2}(B(0,1))'$. □

Theorem 8. *Let $\{X_s\}_{s\in\mathbf{R}^n}$ be an n-dimensional Levy FBM with Hurst parameter $H \in (0,1)$. For any ball $B \subset \mathbf{R}^n$ let E_B be the L^2-space generated by the differences $\{X_{s_1} - X_{s_2} \,|\, s_1, s_2 \in B\}$. Then, if $s_1 \neq s_2$ the subspaces $E_{B(s_1,\varepsilon)}$ and $E_{B(s_2,\varepsilon)}$ are asymptotically independent as $\varepsilon \to 0$. Moreover, there are positive constants $c_1, c_2 > 0$ such that*

$$c_1\varepsilon^{2H-2} \leq \cos(\sphericalangle(E_{B(s_1,\varepsilon)}, E_{B(s_2,\varepsilon)})) \leq c_2\varepsilon^{2H-2}.$$

Proof. The proof is analogous to the proof of Theorem 4. First of all, the lower bound is an immediate consequence of the one-dimensional case since the restriction of the process to a line through the points s_1, s_2 is a one-dimensional FBM. In order to deduce the upper bound we observe that according to Lemma 4 and an easy analogue of Lemma 3 the cosine of the angle between the spaces is given by the quantity

$$A := -\frac{1}{2}\sup_{\phi,\psi} \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} |u-s|^{2H}\phi(u)\overline{\psi(s)}\, duds,$$

where the supremum is taken over all functions $\phi \in C_0^\infty(B(0,1)) \cap W_0^{-n/2-H,2}(B(0,1))$ and $\psi \in C_0^\infty(B(ke_1,1)) \cap W_0^{-n/2-H,2}(B(ke_1,1))$, with unit norm and zero mean. Here $k = |s_1 - s_2|/\varepsilon > 0$. Observe that we used the obvious scaling and rotation invariance of the Levy FBM. By a twofold application of Lemma 5 it follows that

$$A \lesssim \sup_{u\in B(0,1), s\in B(ke_1,1)} \sum_{1\leq|\alpha|\leq n+1, 1\leq|\beta|\leq n+1} \big|D_u^\alpha D_s^\beta\big(|u-s|^{2H}\big)\big| \sim k^{2H-2}.$$

□

Our results raise several interesting open problems related to local independence of stochastic processes. We expect that the methods of the present paper are pretty much restricted to dealing with the FBM, although they may help in obtaining insights and conjectures regarding the following questions.

Q.1 Let $X = \{X_t\}_{t\in\mathbf{R}}$ be a Gaussian process with continuous paths and stationary increments. Find necessary and sufficient conditions for the local independence property, e.g. in terms of the spectral measure of X, or in terms of the variance function $v(t) = \mathbf{E}\,X_t^2$.

With regards to Question 1, we can note a couple of obvious obstacles for local independence. First, if the process is L^2-differentiable, the value of the derivative process belongs to the infinitesimal sigma-algebra around a point (see [16]), and apart from trivial cases this will destroy local independence. Second, periodic processes, like the periodic Brownian bridge defined by the variance function

$$v(t) = \mathbf{E}\,X_t^2 = (t \mod 1)(1-(t \mod 1)),$$

clearly do not satisfy local independence for all times. Periodic components are reflected as atoms of the spectral measure. But are non-smoothness and continuity of spectrum already sufficient for local independence?

One can also ask for a local characterization:

Q.2 Let (X_t) again be a Gaussian process with stationary increments. Give conditions on the variance function $v(t)$ in a neighbourhood of the origin and in a neighbourhood of the point $|t_1 - t_2|$ that would guarantee local independence with respect to points t_1, t_2.

Q.3 Superposing Brownian bridges with different periods, one can probably build examples of non-smooth processes where local independence breaks over any rational distance. But is it possible to construct a continuous but non-differentiable Gaussian process with stationary increments that does not possess local independence over any distance?

Q.4 So far we have only focused on Gaussian processes. Our information-based definition of local independence is, however, meaningful for any kind of stochastic process. It is then interesting to ask about the local independence of various dependent processes. For example, do fractional Lévy processes have this property?

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